Definition: Two events E and F are ***mutually exclusive*** if they can never occur together. In this case $P(E \text{ and } F) = 0$.

The addition rule.

If E and F are mutually exclusive events, then

$$P(E \text{ or } F) = P(E) + P(F).$$

If the events are *not* mutually exclusive, this rule needs to be modified...

If E and F are any two events, then

$$P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$$

Explanation: The sum $P(E) + P(F)$ includes the probabilities of all outcomes that result in E or in F. If E and F are *not* mutually exclusive, some outcomes result in *both* E and F, and these outcomes are counted twice in the sum $P(E) + P(F)$. To correct this, we subtract $P(E \text{ and } F)$.

Observation. The second (general) form of the addition rule includes the first one as a special case.

The laws of probability:

For any events E and F...

1. $0 \leq P(E) \leq 1$.

(*) $P(E) = 1$ means that E is certain to happen.

(*) $P(E) = 0$ means that E is certain *not* to happen.

2. $P(E \text{ and } F) = P(E) \cdot P(F|E)$.

2a. If E and F are ***independent***, then $P(E \text{ and } F) = P(E) \cdot P(F)$.

3. $P(E \text{ or } F) = P(E) + P(F) - P(E \text{ and } F)$.

3a. If E and F are ***mutually exclusive***, then $P(E \text{ or } F) = P(E) + P(F)$

3b. The events E and $E' =$ *not* E are mutually exclusive so

$$P(E) + P(E') = P(E \text{ or } E') = 1 \qquad \Longrightarrow \qquad P(E) = 1 - P(E').$$

Comments:

- The concepts of *independent events* and *mutually exclusive events* are often confused with each other at first. Perhaps because they both sound as if the are describing events that are unrelated to each other.
  - So, to reiterate, E and F are ***independent*** if knowing that one has occurred ***does not change the probability*** of the other.
  - On the other hand, E and F are ***mutually exclusive*** if knowing that one has occurred ***reduces the probability of the other to 0***. This is about as far from *independent* as you can get.
- Rule **3a.** above has a very useful generalization. If E and F are any two events, then on the one hand (E and F) and (E and F') are mutually exclusive, and on the other

$$E = (E \text{ and } F) \text{ or } (E \text{ and } F').$$

It follows then that

$$P(E) = P(E \text{ and } F) + P(E \text{ and } F').$$

The paradox of the Chevalier de Méré

Question: Which event is more likely: rolling at least one ***ace*** in 4 rolls of a single die, or rolling at least one ***pair of aces*** in 24 rolls of a pair of dice?

The Chevalier de Méré thought that both events were equally likely, but noticed that the first seemed to occur slightly more often than the second. His appeal to Blaise Pascal to resolve this puzzle, and the subsequent correspondence between Pascal and Pierre de Fermat on the subject marks the beginning of the modern theory of probability.

Answer: This is an example of a situation where it is easier to compute the probability that an event ***does not*** occur (and use 3b) than to directly compute the probability that the event does occur.

- The probability of ***not*** rolling an ace in one roll is 5/6, and since different rolls are *independent* of each other,

$$P(\text{no aces in 4 rolls}) = \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} = \left(\frac{5}{6}\right)^4 \approx 0.482.$$

  So $P(\text{at least one ace in 4 rolls}) = 1 - P(\text{no aces in 4 rolls}) \approx 0.518.$

- The probability of ***not*** rolling a pair of aces in one roll (of two dice) is 35/36, and since different rolls are *independent* of each other,

$$P(\text{no pair of aces in 24 rolls}) = \overbrace{\frac{35}{36} \cdot \frac{35}{36} \cdots \frac{35}{36}}^{24} = \left(\frac{35}{36}\right)^{24} \approx 0.5086.$$

  So $P(\text{at least one pair of aces in 24 rolls}) \approx 1 - 0.5086 = 0.4914.$

Example. Consider the box $\boxed{1}\ \boxed{2}\ \boxed{2}\ \boxed{4}\ \boxed{6}$.

(*) Two tickets are drawn at random with replacement. What are the chances that both of them are $\boxed{2}$s?

$\Rightarrow$ "With replacement" means that the draws are independent, so...

$$P\left(\boxed{2} \text{ on 1st draw } \textit{and } \boxed{2} \text{ on 2nd draw}\right) = \frac{2}{5} \cdot \frac{2}{5} = 16\%.$$

(*) Two tickets are drawn at random with replacement. What are the chances that exactly one of them is a $\boxed{2}$?

$\Rightarrow$ The event $E =$"exactly one $\boxed{2}$ in two draws" can occur in two *mutually exclusive* ways: $A =$"$\boxed{2}$ on the first draw and something else on the second draw" ***or*** $B =$"something else on the first draw and a $\boxed{2}$ on the second draw". So...

$$P(E) = P(A \text{ or } B) = P(A) + P(B) = \frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5} = 48\%$$

(*) Two tickets are drawn at random with replacement. What are the chances that at least one of them is a $\boxed{2}$?

$\Rightarrow$ "At least one $\boxed{2}$" means either one $\boxed{2}$ or two $\boxed{2}$s in this case. The probability of (exactly) one $\boxed{2}$ is 48% and the probability of two $\boxed{2}$s is 16% (and these are mutually exclusive events), so...

$P(\text{at least one } \boxed{2}) = 48\% + 16\% = 64\%.$

(*) Two tickets are drawn from the box *without* replacement. What are the chances that at least one of them is a $\boxed{2}$?

$\Rightarrow$ We can repeat the logic from above, but we have to adjust the calculations because the draws are no longer independent.

(i) $P(\text{two } \boxed{2}\text{s}) = \frac{2}{5} \cdot \frac{1}{4} = 10\%$

(ii) $P(\text{exactly one } \boxed{2}) = \frac{2}{5} \cdot \frac{3}{4} + \frac{3}{5} \cdot \frac{2}{4} = 60\%$

(iii) $P(\text{at least one } \boxed{2}) = P(\text{exactly one } \boxed{2}) + P(\text{two } \boxed{2}\text{s}) = 70\%.$

We can also answer the last question using a different approach, by considering all the different pairs of tickets we might draw (without replacement) and find the proportion of these pairs that include at least one $\boxed{2}$.

(i) There are 5 ways to choose the first ticket in the pair, and for each choice of first ticket there are 4 ways to choose the second ticket. I.e., there are $5 \cdot 4 = 20$ possible pairs we can draw.

(ii) There are 3 ways to choose a first ticket that is *not* a $\boxed{2}$, and for each such choice there are 2 ways to choose a second ticket that is also *not* a $\boxed{2}$. So there are $3 \cdot 2 = 6$ pairs that we can draw that have *no* $\boxed{2}$s.

(iii) This means that the remaining $14 = 20 - 6$ possible pairs all have at least one $\boxed{2}$, so the chance of observing at least one $\boxed{2}$ (when drawing two tickets without replacement) is $14/20 = 70\%$.

(*) Same box... If 4 tickets are drawn with replacement, what are the chances that we observe the sequence $\boxed{2}$ - $\boxed{1}$ - $\boxed{4}$ - $\boxed{2}$?

$\Rightarrow$ The draws are independent, so

$$P(\boxed{2} - \boxed{1} - \boxed{4} - \boxed{2}) = \frac{2}{5} \cdot \frac{1}{5} \cdot \frac{1}{5} \cdot \frac{2}{5} = 0.64\%$$

(*) If 4 tickets are drawn with replacement, what are the chances that we observe the sequence $\boxed{2}$ - $\boxed{\textit{not}\, 2}$ - $\boxed{\textit{not}\, 2}$ - $\boxed{2}$?

$\Rightarrow$ The draws are still independent, so

$$P(\boxed{2} - \boxed{\textit{not}\, 2} - \boxed{\textit{not}\, 2} - \boxed{2}) = \frac{2}{5} \cdot \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{2}{5} = 5.76\%$$

(*) If 4 tickets are drawn with replacement, what are the chances that we observe *exactly* two $\boxed{2}$ s?

$\Rightarrow$ *'Exactly two'* $\boxed{2}$ s in a sequence of four draws can occur in many ways. For example, ($\boxed{2}$ - $\boxed{\textit{not}\,2}$ - $\boxed{\textit{not}\,2}$ - $\boxed{2}$), ($\boxed{2}$ - $\boxed{2}$ - $\boxed{\textit{not}\,2}$ - $\boxed{\textit{not}\,2}$), ($\boxed{2}$ - $\boxed{\textit{not}\,2}$ - $\boxed{2}$ - $\boxed{\textit{not}\,2}$), and so on.

Two key observations:

(i) All these different sequences are mutually exclusive of each other. This is because, if we observe the sequence ($\boxed{2}$ - $\boxed{\textit{not}\,2}$ - $\boxed{2}$ - $\boxed{\textit{not}\,2}$), for example, then we do not observe the sequence ($\boxed{2}$ - $\boxed{\textit{not}\,2}$ - $\boxed{\textit{not}\,2}$ - $\boxed{2}$).

(ii) The probability of observing each of these sequences is *the same* for all of them, because

$$\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{3}{5}\cdot\frac{2}{5}=\frac{2}{5}\cdot\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{3}{5}=\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{2}{5}\cdot\frac{3}{5}=\cdots=5.76\%$$

This means that

$$P(\text{exactly two } \boxed{2}\text{s in four draws}) = \overbrace{\frac{36}{625} + \frac{36}{625} + \frac{36}{625} + \cdots + \frac{36}{625}}^{\text{number of sequences with two } \boxed{2}\text{s}}$$

The only thing that remains is to figure out how many sequences there are with exactly two $\boxed{2}$s...

To that end, here are a few simplifying observations.

(i) We don't care which tickets go in the 'not $\boxed{2}$' spots.

(ii) Since we are (theoretically) listing **all** of the possible 2-$\boxed{2}$ sequences, we don't need to think about this process as a bunch of 'random draws'... we can be methodical.

(iii) When listing different 2-$\boxed{2}$ sequences, all we have to decide is where in each sequence to put the $\boxed{2}$s... the 'not $\boxed{2}$'s will go in the other two spots.

$\Rightarrow$ *The number of different sequences with two* $\boxed{2}$ *s is equal to the number of ways to choose two positions in a sequence of four.*

$\Rightarrow$ There are 4 positions in which we can place the first $\boxed{2}$, and for each choice of first position, there are 3 ways to choose the second position...

So it seems that there are $4 \cdot 3 = 12$ ways to place two $\boxed{2}$s in a sequence of four draws...

But we are *overcounting*, because each pair of positions has been counted *twice*! For example, the choices *'first* $\boxed{2}$ *in the third position and second* $\boxed{2}$ *in the first position'* and *'first* $\boxed{2}$ *in the first position and second* $\boxed{2}$ *in the third position'* result in the *same* pair of positions — first and third.

Conclusion: The number of sequences with exactly two $\boxed{2}$s is $\dfrac{4 \cdot 3}{2} = 6$... So

$$P(\text{exactly two } \boxed{2}\text{s in four draws}) = \overbrace{\frac{36}{625} + \frac{36}{625} + \frac{36}{625} + \cdots + \frac{36}{625}}^{\text{number of sequences with two } \boxed{2}\text{s}}$$

$$= \left(\frac{36}{625}\right) \times 6 = \frac{216}{625} = 34.56\%$$

Bonus content:

Chapter 14, review problem 13:

If you draw n marbles from the box at random with replacement, then

$$P(\textbf{\textit{no}}\text{ red marbles drawn in } n \text{ draws}) = \overbrace{(0.98)\cdot(0.98)\cdots(0.98)}^{n} = (0.98)^n$$

and therefore

$$P(\textbf{\textit{at least one}}\text{ red marble drawn in } n \text{ draws}) = 1 - (0.98)^n.$$

Now, we want to find n so that $1 - (0.98)^n > 0.5$ which is the same as $(0.98)^n < 0.5$.

Approach #1. Use a calculator to find $(0.98)^n$ for $n = 1, 2, \ldots$ and stop when you first go below 0.5.

Approach #2. Solve $(0.98)^x = 0.5$:

$$\Longrightarrow\ x\ln(0.98) = \ln(0.5)\ \Longrightarrow\ x = \frac{\ln 0.5}{\ln 0.98} \approx 34.3.$$

This means that $n = 35$ is the number that we want.

Chapter 14, review problem 14:

Since both players have two *different* tickets each, they both have the same chance of winning (about $\frac{1}{11478740}$).

Just because the second pair of tickets are more different from each other than the first pair doesn't change the rules of the game. You don't win if you match more of the numbers than your friend, you only win if you match all six numbers exactly on one ticket.

In other words the ticket 1 2 3 4 5 6 is just as likely to win as the ticket 4 41 23 7 19 32, or any other ticket. And likewise, any pair of *different* tickets (e.g., 1 2 3 4 5 6 and 2 3 4 5 6 7) is just as likely to win as any other pair of different tickets.